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Convergence of Best Approximations in a Smooth Banach Space

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Let X be a reflexive, strictly convex Banach space such that both X and X^* have Fréchet differentiable norms, and let $\{C_n\}$ be a sequence of non-empty closed convex subsets of X. We prove that the sequence of best approximations $\{p(x \mid C_n)\}$ of any $x \in X$ converges if and only if $\lim C_n$ exists and is not empty. We also discuss measurability of closed convex set valued functions.

0. INTRODUCTION

Let X be a Banach space. If X is reflexive and strictly convex, then for any non-empty closed convex subset C of X and $x \in X$ there exists a unique best approximation $p(x \mid C)$ of x in C. If every sequence $\{x_n\} \subset X$ which weakly converges to some $x \in X$ and satisfies $\|x_n\| \to \|x\|$ as $n \to \infty$ necessarily converges to x in the norm, we say that X has Property (H). If X is reflexive, strictly convex and has Property (H), $x \mapsto p(x \mid C)$ is norm-to-norm continuous. In this paper we investigate continuity of $C \mapsto p(x \mid C)$. This was first considered by Brosowski, Deutsch and Nürnberger [1]. They considered a family $\{V_a\}$ of subsets of normed linear space X parametrized by a topological space and studied continuity of multivalued mappings $a \mapsto V_a$ and $a \mapsto P(x \mid V_a)$. $P(x \mid V_a)$ is the set of best approximations of x in V_a. On the other hand, our method is not parametrized.

Let $\{C_n\}$ be a sequence of non-empty closed convex subsets of X. Mosco [8] defined $\lim C_n$. We prove that if X is reflexive and strictly convex and has Property (H), then for any $x \in X$ the sequence of best approximations $\{p(x \mid C_n)\}$ converges whenever $\lim C_n$ exists and is not empty. This was proved by Rao [9] in which $\{C_n\}$ is increasing with respect to set inclusion. Conversely, if X has a Fréchet differentiable norm, then $\lim C_n$ exists and is not empty whenever the sequence $\{p(x \mid C_n)\}$ of best approximations converges for every $x \in X$. Since the condition that X is reflexive and strictly

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convex and has Property (H) is equivalent to that X^* has the Fréchet differentiable norm, if both X and X^* have the Fréchet differentiable norms, the sequence $\{p(x \mid C_n)\}$ of best approximations of any $x \in X$ converges if and only if $\lim C_n$ exists and is not empty. If X is an L^p-space $(1 < p < \infty)$, it is the case that the above assertion is valid. The author [11] has proved it in which X is a Hilbert space and investigated the limit of σ-fields in probability measure spaces.

In the last section we define strong measurability of closed convex set valued functions. In a certain Banach space it is equivalent to some measurability conditions defined by Himmerberg [6].

1. Notations

Let X be a Banach space with norm $\|\cdot\|$, S be the closed unit ball of X, and $\mathfrak{C}$ be the set of all, non-empty, closed convex subsets of X.

For any $x \in X$ and $A \subset X$ we define $d(x, A) = \inf_{y \in A} \|x - y\|$ and $P(x \mid A) = \{y \in A : \|x - y\| = d(x, A)\}$.

Remark (see, for example, Singer [10]). (i) Let $\varnothing \neq A \subset X$. Then $|d(x, A) - d(y, A)| \leqslant \|x - y\|$ for any x, $y \in X$. In particular, $d(\cdot, A)$ is uniformly continuous;

(ii) $P(x \mid C)$ consists of at most single element for any $x \in X$ and $C \in \mathfrak{C}$ if and only if X is strictly convex (i.e., S is strictly convex);

(iii) $P(x \mid C)$ is not a void set for any $x \in X$ and $C \in \mathfrak{C}$ if and only if X is reflexive.

We consider some properties for X. Notations are due to Cudia [3] and Day [4].

(H) If a sequence $\{x_n\} \subset X$ weakly converges to $x \in X$ and $\|x_n\| \to \|x\|$ as $n \to \infty$, then $\|x - x_n\| \to 0$ as $n \to \infty$.

(K) For any $C \in \mathfrak{C}$ the diameter of $(C \cap r \cdot S)$ tends to 0 as $r \to d(0, C)$.

(F) X has the Fréchet differentiable norm, i.e., for any $x \in S$ there exists $\lim_{t \to 0} (\|x + t \cdot y\| - \|x\|)/t$ uniformly for $y \in S$.

The following assertions are equivalent (see Rao [9]):

(i) X is strictly convex, reflexive, and has (H);

(ii) X has (K);

(iii) X^* has the Fréchet differentiable norm.

If X is strictly convex and reflexive, we denote by $p(x \mid C)$ the unique element of $P(x \mid C)$ for any $x \in X$ and $C \in \mathfrak{C}$. Then $p(\cdot \mid C)$ is norm-to-weak continuous. Moreover if X has (H) (hence X has (K)), it is norm-to-norm continuous (see Singer [9]).

2. The Limit of a Sequence of Closed Convex Sets

Let $\{C_n\}$ be a sequence in $\mathfrak{C}$. Mosco [8] defined a *strong lower limit* s-lim inf C_n as the set of all $x \in X$ such that there exist $x_n \in C_n$ for almost all n and it tends to x as $n \to \infty$ in the norm, and a *weak upper limit* w-lim sup C_n as the set of all $x \in X$ such that there exist a subsequence $\{C_{n'}\}$ of $\{C_n\}$ and $x_{n'} \in C_{n'}$ for every n' and it tends to x as $n' \to \infty$ in the weak topology. The *weak lower limit* w-lim inf C_n and the *strong upper limit* s-lim sup C_n are defined similarly, but we do not use them in this paper. If s-lim inf $C_n =$ w-lim sup C_n, then the common value is denoted by lim C_n, and in this case all of the limits defined above coincide. The following proposition is a direct consequence of the definition, and some other elementary properties and examples are discussed in [8].

Proposition 2.1. *Let $\{C_n\}$ be a sequence in $\mathfrak{C}$.*

(i) s-lim inf $C_n = \{x \in X: d(x, C_n) \to 0$ *as* $n \to \infty\}$;

(ii) s-lim inf $C_n \in \mathfrak{C} \cup \{\varnothing\}$;

(iii) $\overline{\bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} C_n} \subset$ s-lim inf $C_n \subset$ w-lim sup $C_n \subset \bigcap_{m=1}^{\infty} \overline{\text{co}} \bigcup_{n=m}^{\infty} C_n$, *where* $\overline{\text{co}}$ *means the closed convex hull.*

Theorem 2.2. *Let $\{C_n\}$ be a sequence in $\mathfrak{C}$.*

(i) lim sup $d(x, C_n) \leqslant d(x,$ s-lim inf $C_n)$ *for every* $x \in X$.

(ii) *If* $C \in \mathfrak{C}$ *satisfies* lim sup $d(x, C_n) \leqslant d(x, C)$ *for every* $x \in X$, *then* $C \subset$ s-lim inf C_n.

We assume further X to be reflexive.

(iii) lim inf $d(x, C_n) \geqslant d(x,$ w-lim sup $C_n)$ *for every* $x \in X$.

(iv) *If X is finite dimensional or has* (F), *and if* $C \in \mathfrak{C}$ *satisfies* lim inf $d(x, C_n) \geqslant d(x, C)$ *for every* $x \in X$, *then* $C \supset$ w-lim sup C_n.

Proof. (i) Let $x \in X$ be fixed. For any $y \in$ s-lim inf C_n there exists a sequence $\{y_n\}$ such that $y_n \to y$ as $n \to \infty$ and $y_n \in C_n$ for every n. Hence

$$\limsup d(x, C_n) \leqslant \lim \|x - y_n\| = \|x - y\|,$$

for any $y \in$ s-lim inf C_n. Thus lim sup $d(x, C_n) \leqslant d(x,$ s-lim inf $C_n)$.

(ii) Let $C \in \mathfrak{C}$ satisfy $\limsup d(x, C_n) \leqslant d(x, C)$ for every $x \in X$. Then for any $x \in C$, $\limsup d(x, C_n) = 0$. Therefore by Proposition 2.1 we have the wanted result.

(iii) Assume that there exists $x \in X$ such that $\liminf d(x, C_n) > d(x, \text{w-}\limsup C_n)$. Let $\{C_{n'}\}$ be a subsequence of $\{C_n\}$ with $\lim d(x, C_{n'}) = \liminf d(x, C_n)$. Since X is reflexive, there exists $x_{n'} \in P(x \mid C_{n'})$ for every n'. Then $\{x_{n'}\}$ is norm bounded because $\lim d(x, C_{n'}) < \infty$. Hence by the Banach–Alaoglu and Eberein–Šmulian theorems there exists a subsequence $\{x_{n''}\}$ of $\{x_{n'}\}$ which weakly converges to some $x' \in X$. Then $x' \in \text{w-}\limsup C_n$. Therefore we have

$$d(x, \text{w-}\limsup C_n) > \liminf d(x, C_n) = \lim \|x - x_{n''}\| \geqslant \|x - x'\|,$$

where the last inequality follows from weak lower semicontinuity of norm $\|\cdot\|$. This is a contradiction. Hence we have (iii).

Before proving (iv), we show some technical lemmas.

LEMMA 2.3. *Let X be reflexive and $C \in \mathfrak{C}$ with $0 \notin C$. Then $x \in P(-\lambda \cdot x \mid C)$ for any $x \in P(0 \mid C)$ and $\lambda \geqslant -1$.*

Proof. Let $x \in P(0 \mid C)$. By Theorem 1.1 of Singer [10, p. 360], there exists $f \in X^*$ such that $\|f\| = 1$ and $\operatorname{Re} f(y) \geqslant \|x\|$ for any $y \in C$. Then

$$\begin{aligned} \operatorname{Re} f(y + \lambda \cdot x) &= \operatorname{Re} f(y) + \lambda \cdot \operatorname{Re} f(x) \\ &\geqslant (1 + \lambda) \cdot \|x\| = \|x + \lambda \cdot x\|, \end{aligned}$$

for any $y \in C$ and $\lambda \geqslant -1$. Using once more the theorem mentioned above, we have the lemma.

LEMMA 2.4. *If X has (F), for any $x \in X \setminus \{0\}$ and a sequence $\{x_n\} \subset X$ which weakly converges to 0 there exists $0 \leqslant \theta < 1$ such that* $\liminf_n \|\theta \cdot x + (1 - \theta) \cdot x_n\| < \|x\|$.

Proof. We may assume $\|x\| = 1$ without loss of generality. Since X has (F), X is smooth. Hence there uniquely exists $f \in X^*$ with $\|f\| = f(x) = 1$. Then for any $y \in X$ with $\operatorname{Re} f(y) < 1$ the line segment $[x, y]$ intersects to $S \setminus \{x\}$. (This follows from the Hahn–Banach theorem.) Since $\{x_n\}$ weakly converges to 0, we may assume that $\operatorname{Re} f(x_n) < 1$ for every n. On the other hand, if $\|x_n\| < 1$ for infinitely many n, the lemma is trivial. Hence we may assume that $\|x_n\| \geqslant 1$ for every n. Therefore there exists $y_n \in (x, x_n]$ with $\|y_n\| = 1$. Then $\liminf \|x - y_n\| > 0$. If it is not true, there exists a subsequence $\{y_{n'}\}$ of $\{y_n\}$ such that $\lim \|x - y_{n'}\| = 0$. Hence, by the Fréchet differentiability of the norm $\|\cdot\|$, it follows that

$$\frac{1-f(x_{n'})}{\|x-x_{n'}\|}=\frac{1-f(y_{n'})}{\|x-y_{n'}\|}$$
$$=\frac{\|x+(y_{n'}-x)\|-\|y_{n'}\|-f(y_{n'}-x)}{\|y_{n'}-x\|}\to 0$$

as $n\to\infty$. Since $1-f(x_{n'})\to 1$ as $n\to\infty$, $\lim\|x-x_{n'}\|=\infty$. But, since $\{x-x_n\}$ is a weak converning sequence, by the uniform boundedness theorem we have $\sup_n\|x-x_n\|<\infty$. This is a contradiction. Thus putting

$$m=\liminf\|x-y_n\|,$$
$$M=\sup\|x-x_n\|,$$
$$\theta=1-m/(2\cdot M),$$

we have the lemma.

Proof of (iv). Let $C\in\mathfrak{C}$ satisfy $\liminf d(x, C_n)\geqslant d(x, C)$ for every $x\in X$. We fix any $y\in$ w-lim sup C_n. Then there exists a subsequence $\{C_{n'}\}$ of $\{C_n\}$ and $y_{n'}\in C_{n'}$ for every n' such that $\{y_{n'}\}$ weakly converges to y. We shall show that y belongs to C. By the parallel translation, it may be assumed that $y=0$. Hence it suffices to show that $0\in C$. Now assume that $0\notin C$, and take $z\in P(0\mid C)$. Then $z\neq 0$. Since for any $x\in X$

$$\liminf\|x-y_{n'}\|\geqslant\liminf d(x, C_{n'})\geqslant\liminf d(x, C_n)\geqslant d(x, C)$$

and by Lemma 2.3, $z=P(-\lambda\cdot z\mid C)$ for any $\lambda\geqslant 0$, we have

$$\liminf\|\lambda\cdot z+y_{n'}\|\geqslant d(-\lambda\cdot z, C)=(\lambda+1)\cdot\|z\|$$

for any $\lambda\geqslant 0$. Dividing both sides by $1+\lambda$, we have

$$\liminf\|\theta\cdot z+(1-\theta)\cdot y_{n'}\|\geqslant\|z\|$$

for any $0\leqslant\theta<1$. If X is finite dimensional, since $\{y_{n'}\}$ strongly converges to 0, this is a contradiction. On the other hand, if X has (F), this also contradicts Lemma 2.4. Thus we have $0\in C$.

THEOREM 2.5. *Let X be reflexive and $\{C_n\}$ be a sequence in $\mathfrak{C}$.*

(i) *If* $\lim C_n$ *exists, then* $d(x, C_n)$ *tends to* $d(x, \lim C_n)$ *as* $n\to\infty$ *for every* $x\in X$.

(ii) *If X is finite dimensional or has* (F), *and if there exists* $C\in\mathfrak{C}\cup\{\varnothing\}$ *such that* $d(x, C_n)$ *tends to* $d(x, C)$ *as* $n\to\infty$ *for every* $x\in H$, *then* $\lim C_n=C$.

3. Convergence of Best Approximations

In this section we assume X to be reflexive and strictly convex.

Lemma 3.1. *Let* $\{C_n\}$ *be a sequence in* $\mathfrak{C}$ *such that* s-lim inf $C_n \neq \varnothing$. *Then* $\{p(x \mid C_n)\}$ *is a norm bounded set for any* $x \in X$.

Proof. Let y belong to s-lim inf C_n. Then there exists a sequence $\{y_n\}$ such that $y_n \to y$ as $n \to \infty$ and $y_n \in C_n$ for every n. Since $\sup \|y_n\| = M < \infty$ and for any $x \in H$

$$\begin{aligned}\|p(x \mid C_n)\| &\leqslant \|p(x \mid C_n) - x\| + \|x\| \\ &\leqslant \|y_n - x\| + \|x\| \leqslant M + 2 \cdot \|x\|,\end{aligned}$$

we have the lemma.

Theorem 3.2. *Let* $\{C_n\}$ *be a sequence in* $\mathfrak{C}$.

(i) *If* lim C_n *exists and is not empty, then* $\{p(x \mid C_n)\}$ *weakly conveges to* $p(x \mid \lim C_n)$ *for every* $x \in X$. *Moreover if* X *has* (H), *the convergence is in the norm.*

(ii) *If* X *is finite dimensional or has* (F), *and if* $\{p(x \mid C_n)\}$ *is a norm converging sequence for every* $x \in X$, *then* lim C_n *exists and* $\{p(x \mid C_n)\}$ *converges to* $p(x \mid \lim C_n)$ *for every* $x \in X$.

Proof. (i) Let $x \in X$ be fixed. Since, by Lemma 3.1, $\{p(x \mid C_n)\}$ is norm bounded, for any subsequence $\{p(x \mid C_{n'})\}$ of $\{p(x \mid C_n)\}$, there exists a subsequence $\{p(x \mid C_{n''})\}$ which weakly converges to some $y \in X$. Then $y \in$ w-lim sup $C_n = \lim C_n$. For any $z \in \lim C_n$ we have

$$\begin{aligned}\|x - y\| &\leqslant \liminf \|x - p(x \mid C_{n''})\| \\ &\leqslant \lim \|x - p(z \mid C_{n''})\| = \|x - z\|.\end{aligned}$$

Therefore $y = p(x \mid \lim C_n)$. Since any subsequence $\{p(x \mid C_{n'})\}$ of $\{p(x \mid C_n)\}$ has a subsequence $\{p(x \mid C_{n''})\}$ which weakly converges to $p(x \mid \lim C_n)$, $\{p(x \mid C_n)\}$ also weakly converges to $p(x \mid \lim C_n)$. On the other hand, by Theorem 2.5, $\|x - p(x \mid C_n)\| \to \|x - p(x \mid \lim C_n)\|$ as $n \to \infty$. Therefore, if X has (H), $\{p(x \mid C_n)\}$ converges to $p(x \mid \lim C_n)$.

(ii) We put $C =$ s-lim inf C_n. If $\{p(x \mid C_n)\}$ converges to $y \in X$, then it follows that for any $z \in C$

$$\begin{aligned}\|x - y\| &= \lim \|x - p(x \mid C_n)\| \\ &\leqslant \lim \|x - p(z \mid C_n)\| = \|x - z\|,\end{aligned}$$

and that $y = p(x \mid C)$. Hence $d(x, C_n)$ tends to $d(x, C)$ as $n \to \infty$. By Theorem 2.5(ii), we have $\lim C_n = C$.

THEOREM 3.3. *Suppose that X satisfies one of the following conditions:* (A) *X is finite dimensional and strictly convex*; *or* (B) *both X and X^* have Fréchet differentiable norms. Then for any sequence $\{C_n\} \subset \mathfrak{C}$ the following assertions are equivalent*:

(i) *$\lim C_n$ exists and is not empty*;

(ii) *there exists $C \in \mathfrak{C}$ such that $d(x, C_n)$ tends to $d(x, C)$ as $n \to \infty$ for every $x \in X$*;

(iii) *$\{p(x \mid C_n)\}$ is a norm convergent sequence for every $x \in X$.*

Proof. This is the direct consequence of Theorems 2.5 and 3.2.

4. Measurability of Closed Convex Set Valued Functions

In this section let X be a separable Banach space and satisfy condition (A) or (B) in Theorem 3.3.

THEOREM 4.1. *$\mathfrak{C}$ has a separable metric δ such that $\delta(C_n, C) \to 0$ as $n \to \infty$ if and only if* $\lim C_n = C \neq \varnothing$.

Proof. Let $D = \{x_k\}$ be a countable dense subset of X. We define $\delta_k(C_1, C_2) = |d(x_k, C_1) - d(x_k, C_2)|$ for every C_1, $C_2 \in \mathfrak{C}$ and k. Then $\{\delta_k\}$ is a family of semimetrics on X and separates points of X. Moreover, by Theorem 2.5 we have that $\delta_k(C_n, C) \to 0$ as $n \to \infty$ for every k if and only if $\lim C_n = C \neq \varnothing$. Therefore, putting

$$\delta(C_1, C_2) = \sum_{k=1}^{\infty} \frac{\delta_k(C_1, C_2)}{2^k \cdot (1 + \delta_k(C_1, C_2))},$$

for any C_1, $C_2 \in \mathfrak{C}$, we have the theorem. Separability is proved as follows. We define $\mathfrak{C}_D = \{\overline{\mathrm{co}}\,\{x_{k_1}, ..., x_{k_n}\}\colon x_{k_1}, ..., x_{k_n} \in D\}$. Let any $C \in \mathfrak{C}$ be fixed. For any n we define $C_n = \{x \in X\colon d(x, C) \leqslant 1/n\}$. Then we easily see that $C_n \in \mathfrak{C}$ for every n and $C_n \downarrow C$ as $n \to \infty$. Hence we have that $\lim C_n = C$ and that $\delta(C_n, C) \to 0$ as $n \to \infty$. On the other hand, for any n let $C_n \cap D = \{x_{k_1}, x_{k_2}, ...\}$ and $C_n^m = \{x_{k_1}, ..., x_{k_n}\}$ for every m. Since $\overline{C_n \cap D} = C_n$, $C_n^m \uparrow C_n$ as $m \to \infty$ for every n. Hence we have that $\lim_m C_n^m = C_n$ and that $\delta(C_n^m, C_n) \to 0$ as $m \to \infty$ for every n. Therefore for any $\varepsilon > 0$ we can find C_n^m with $\delta(C_n^m, C) \leqslant \varepsilon$. Thus $\mathfrak{C}_D$ is a countable dense subset of $\mathfrak{C}$.

Remark. Let $\mathfrak{C}^b$ be the set of all norm bounded elements of $\mathfrak{C}$. It is well known that $\mathfrak{C}^b$ has the so-called *Hausdorff metric*. If X is finite dimensional, the topology induced by the Hausdorff metric is equal to the topology induced by δ. But, if X is infinite dimensional, the former is really stronger than the latter (see Mosco [8, Lemma 1.1]).

Let (Ω, Σ) be a measurable space. A function $F: \Omega \to \mathfrak{C}$ is called to be *simple* if there exist a countable partition $\{A_n\} \subset \Sigma$ of Ω and $\{C_n\} \subset \mathfrak{C}$ such that $F(\omega) = C_n$ if $\omega \in A_n$. If there exists a sequence $\{F_n\}$ of simple functions and $F(\omega) = \lim F_n(\omega)$ for any $\omega \in \Omega$, we say F to be *strongly measurable.*

For any function $F: \Omega \to \mathfrak{C} \cup \{\varnothing\}$ we define $D(F) = \{\omega \in \Omega: F(\omega) \neq \varnothing\}$ and call it the *domain* of F.

THEOREM 4.2. *For any function $F: \Omega \to \mathfrak{C} \cup \{\varnothing\}$ the following assertions are equivalent*:

(i) *F is strongly measurable*;

(ii) *$D(F) \in \Sigma$ and $p(x \mid F(\cdot))$ is strongly measurable on $D(F)$ for every $x \in X$ in the sense of Hille and Phillips* [5];

(iii) *$d(x, F(\cdot))$ is measurable for every $x \in X$*;

(iv) *F is $(\Sigma, \mathbf{B}(\mathfrak{C}))$-measurable, where $\mathbf{B}(\mathfrak{C})$ is the Borel field on $\mathfrak{C}$ induced by metric δ.*

Proof. (i) $\Rightarrow$ (ii) Let $\{F_n\}$ be a sequence of simple functions with $F(\omega) = \lim F_n(\omega)$ for any $\omega \in \Omega$. Then by Theorem 2.5(i) we have

$$D(F) = \{\omega \in \Omega: d(0, F(\omega)) \neq \infty\}$$
$$= \{\omega \in \Omega: \lim \| p(0 \mid F_n(\omega))\| \neq \infty\}.$$

Since $\| p(0 \mid F_n(\cdot))\|$ is measurable for every n, it follows that $D(F) \in \Sigma$. By Theorem 3.2 for any $x \in X$ and $\omega \in D(F)$, $\{p(x \mid F_n(\omega))\}$ converges to $p(x \mid F(\omega))$. Since $p(x \mid F_n(\cdot))$ is countably valued for every n, $p(x \mid F(\cdot))$ is strongly measurable.

(ii) $\Rightarrow$ (iii) Since $d(x, F(\omega)) = \|x - p(x \mid F(\omega))\|$ for every $x \in X$ and $\omega \in D(F)$, this is trivial.

(iii) $\Rightarrow$ (iv) Let $G_r(C)$ be any open ball with diameter r and at center $C \in \mathfrak{C}$. Then

$$F^{-1}(G_r(C)) = \left\{\omega \in \Omega: \sum_{k=1}^{\infty} \frac{\delta_k(F(\omega), C)}{2^k \cdot (1 + \delta_k(F(\omega), C))} < r\right\}.$$

By the assumption, $\delta_k(F(\cdot), C)$ is measurable for every k. Hence $F^{-1}(G_r(C)) \in \Sigma$, and we have (iv).

(iv) $\Rightarrow$ (i) Let $\mathfrak{C}_D = \{C_k\}$ be a countable dense subsets of $\mathfrak{C}$. Then for any $\varepsilon > 0$ and $C_k \in \mathfrak{C}_D$ we define $A(\varepsilon, k) = \{\omega \in \Omega: \delta(C_k, F(\omega)) \leqslant \varepsilon\}$. Moreover we define $B(\varepsilon, k) = A(\varepsilon, k) \backslash \bigcup_{i=1}^{k-1} A(\varepsilon, i)$ for every k and

$F_\varepsilon(\omega) = C_k$ if $\omega \in B(\varepsilon, k)$. Then F_ε is a simple function and $\delta(F_\varepsilon(\omega), F(\omega)) \to 0$ as $\varepsilon \to 0$ and $\lim_{\varepsilon \to 0} F_\varepsilon(\omega) = F(\omega)$ for any $\omega \in \Omega$. Thus F is strongly measurable.

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